Journal of Engineering Mathematics, Vol. 11, No. 1, January 1977
Noordhoff International Publishing–Leyden
Printed in The Netherlands

The Stokes flow round a smooth body with an attached vortex

K. B. RANGER

Department of Mathematics, University of Toronto, Toronto, Canada

(Received October 17, 1975)

SUMMARY

Two dimensional Stokes flow is considered round a smooth body containing in general a concave region facing the fluid. It is found that when the profile is shaped in a more complicated manner than a circle or an ellipse an attached vortex is present in the flow and this can exist when the body is partly concave.

1. Introduction

In [1] the streaming Stokes flow past a spherical cap was examined in some detail. The spherical cap (umbrella shape) appears to be the simplest axially symmetric geometrical shape with a concave region facing the fluid for which analytical treatment is possible. It was found for any angle α of the cap $(0 < \alpha < \pi)$ there is a stream surface $\psi = 0$ in the fluid and bounded by the rim of the cap which traps the fluid on the concave surface of the cap in the form of a ring vortex such that $\psi = 0$, separates this vortex from the mainstream motion. In the case of the cap the stresses and vorticity are singular at the rim which separates regions of positive and negative vorticity on the boundary.

The purpose of this paper is to examine the flow past a smooth body containing a concave or re-entry region facing the fluid. The difficulty in this situation is to find a smooth body for which analytical treatment is possible. In fact, for the axially symmetric flow, though it is possible to determine the stream function and velocity field explicitly, the results cannot be found in a closed form suitable for determining any useful physical information concerning the fluid flow. With this in mind, the most tractable approach is to consider a two dimensional flow analogue of the three dimensional motion. There is an affinity between the two dimensional and axially symmetric flows and a physical feature present in one flow is almost certain to be present in the other.

Since it is not possible to find a streaming Stokes flow in two dimensions past a fixed body the leading term in the Stokes inner expansion will be determined analytically. This solution will represent the flow in the neighbourhood of the boundary and becomes infinite most slowly at large distances from the body. The flow at infinity is in fact that due to a Stokeslet situated at the origin. The body considered in this paper is the geometrical inverse of an oblate ellipse with respect to the unit circle centred outside its boundary, and on a line coincident with its minor axis. The inverse boundary lies between the arc of a circle in one limiting case and a complete circle in the other limit situation. For intermediate values of the parameters the boundary is smooth and has a concave or re-entry region facing the fluid. The stream function ψ_i in the inverse plane is found using inversion and a conformal mapping technique for biharmonic functions, and a fairly simple closed form is determined for the complex velocity. The vorticity is found on the boundary and discussed for representative values of the parameters.

It is found that a vortex is attached to the boundary when the profile is concave and that concavity is a necessary condition for this vortex to form. For the case of a circular arc there is a vortex attached to the concave face for all angles of the arc, but in the other limiting case of a circle there is no separation at all.

Finally the axially symmetric flow past a limaçon of revolution, $r = 1 + \varepsilon \cos\theta$, $0 < \varepsilon < 1$, is discussed as a regular perturbation of the sphere in the parameter ε. The two and three term Stokes expansion both predict that a vortex will form when ε is approximately $\frac{1}{2}$. This is the value of ε when the limaçon develops an indentation about its rear end $\theta = \pi$. Since Stokes flow is reversible the vortex will be attached to the boundary in front or behind the body according to the direction of the flow.

2. Flow past a circle

The solution of the biharmonic equation which becomes infinite most slowly at large distances from a circle $r = 1$ is determined from the boundary value problem

$$\nabla_1^4\psi = 0,\ \nabla_1^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{1}$$

$$\psi = \frac{\partial\psi}{\partial r} = 0 \text{ at } r = 1, \quad \psi \sim r\log r \sin\theta \text{ as } r \to \infty. \tag{2}$$

The solution is easily found to be

$$\psi = \left\{r\log r - \tfrac{1}{2}\left(r - \frac{1}{r}\right)\right\}\sin\theta, \tag{3}$$

and the vorticity on the boundary is given by

$$\omega = \nabla_1^2\psi = 2\sin\theta, \quad r = 1,\ 0 \leqq \theta \leqq \pi. \tag{4}$$

Since both the stream function and its normal derivative vanish on the boundary it is necessary only to observe changes in the sign of ω to determine whether reverse flow will occur. In the present case ω is positive for $0 \leqq \theta \leqq \pi$ and there will be no reverse flow. The same remark also applies to a prolate or oblate ellipse but for more complicated shapes like the one treated in the next section the flow is not as simple and reverse flow will occur for certain values of the geometric parameters.

3. Inversion of biharmonic functions

The transformation

$$x_i = \frac{x - c}{R^2},\ y_i = \frac{y}{R^2},\ R_i^2 = x_i^2 + y_i^2, \tag{5}$$

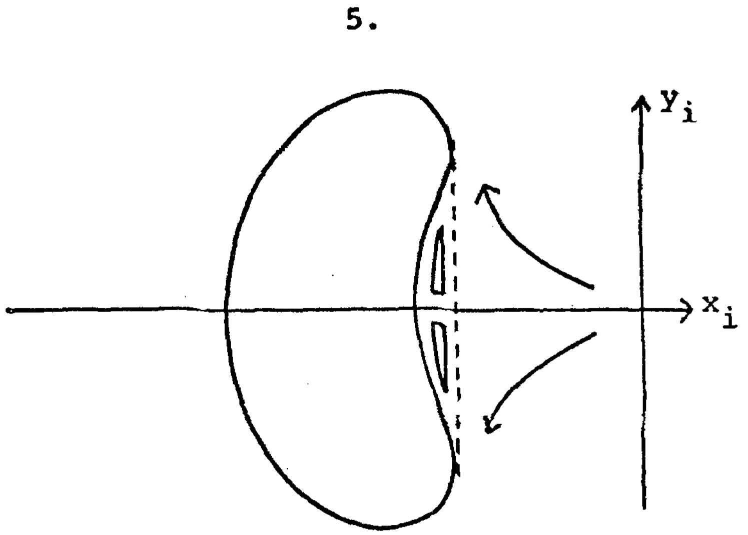

Figure 1. $z_i = x_i + iy_i$ plane.

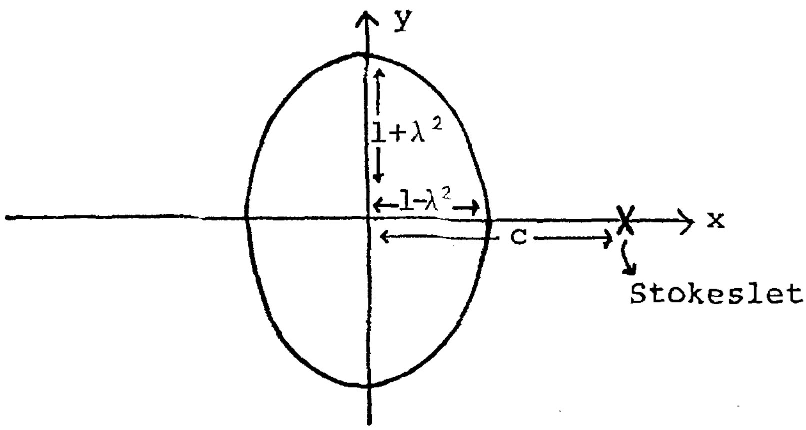

Figure 2. z-plane.

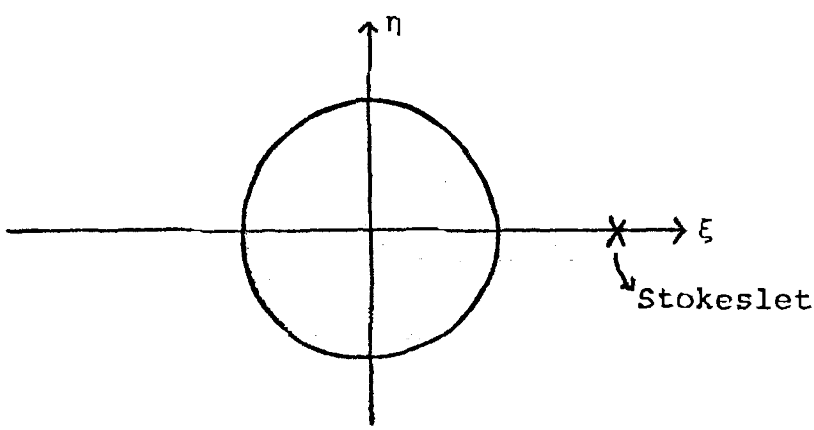

Figure 3. ζ-plane.

where $R = 1/R_i = [(x - c)^2 + y^2]^{\frac{1}{2}}$ defines geometrical inversion with respect to the unit circle centred at the point $(c, 0)$ in the (x, y) plane. If ψ_i is the stream function in the inverse plane it is known that [2]

$$\psi_i = \frac{\psi}{R^2}, \tag{6}$$

where ψ_i is biharmonic in the (x_i, y_i) plane and ψ is biharmonic in the (x, y) plane.

In the present situation the boundary in the (x, y) plane is an oblate ellipse defined by

$$z = x + iy = \zeta - \frac{\lambda^2}{\zeta}, \quad |\lambda| < 1, \tag{7}$$

with $|\zeta| = 1$, on the ellipse. The inverse curve c_i is of fourth order which in general possesses a region which is concave to the fluid region, as shown in the diagram. The inverse transformation of (7) is expressed by

$$\zeta = \frac{z}{2} + \tfrac{1}{2}(z^2 + 4\lambda^2)^{\frac{1}{2}}, \tag{8}$$

so that the exterior of the unit circle $|\zeta| = 1$, maps into the exterior of the ellipse. Furthermore if $z(d) = c$ and $d > 1$, the exterior of the ellipse inverts into the exterior of the closed curve c_i and vice versa. The flow at infinity in the (x_i, y_i) plane is equivalent to a Stokeslet at the origin and is given by

$$\psi_i \sim y_i \log R_i \quad \text{as} \quad R_i \to \infty, \tag{9}$$

so that in the (x, y) plane there is a Stokeslet at $z = c$ and the local flow is expressed by

$$\psi \sim \frac{y_i}{R_i^2} \log R_i = -y \log R \quad \text{as} \quad (x - c)^2 + y^2 \to 0. \tag{10}$$

$$= -\frac{1}{4i}(z - \bar{z})\{\log(z - c) + \log(\bar{z} - c)\}, \tag{11}$$

as $z \to c$. The complex velocity

$$u + iv = 2i\frac{\partial\psi}{\partial\bar{z}} \sim -\tfrac{1}{2}\left\{-\log(z - c) - \log(\bar{z} - c) + (z - \bar{z})\frac{1}{(\bar{z} - c)}\right\} \quad \text{as} \quad z \to c, \tag{12}$$

and in terms of ζ plane coordinates, the singularity in the velocity as $\zeta \to d$ is

$$2i\psi_{\bar{z}} \sim -\tfrac{1}{2}\left\{-\log(\zeta - d) - \log(\bar{\zeta} - d) + \frac{\{z(\zeta) - \bar{z}(\bar{\zeta})\}}{\bar{z}'(d)}\,\frac{1}{(\bar{\zeta} - d)}\right\} \quad \text{as} \quad \zeta \to d, \tag{13}$$

On the boundary of the unit circle in the ζ plane $\zeta = e^{i\phi}$, the stream function ψ satisfies the boundary conditions

$$\psi = \frac{\partial\psi}{\partial\bar{\zeta}} = 0, \quad \text{on} \quad \zeta = e^{i\phi}. \tag{14}$$

The factor $-\frac{1}{2}$ will be removed from (13) and taken as unity in the analysis which follows.

Let $\Omega = \phi + i\psi$ and Ω satisfy

$$\frac{\partial^2 \Omega}{\partial \bar{z}^2} = 0, \tag{15}$$

then $\Omega(z, \bar{z}) = F(z) + \bar{z}G(z)$, where F and G are arbitrary functions of z. ϕ and ψ are conjugate biharmonic functions defined by

$$\left.\begin{aligned} 2\phi &= F(z) + \bar{z}G(z) + \bar{F}(\bar{z}) + z\bar{G}(\bar{z}), \\ 2i\psi &= F(z) - \bar{F}(\bar{z}) + \bar{z}G(z) - z\bar{G}(\bar{z}). \end{aligned}\right\} \tag{16}$$

If $z \equiv z(\zeta)$ is a conformal transformation which maps the unit circle in the ζ-plane into the boundary c in the (x, y) plane, ψ can be represented by

$$2i\psi = f(\zeta) - \bar{f}(\bar{\zeta}) - g(\zeta)\bar{z}(\bar{\zeta}) + \bar{g}(\bar{\zeta})z(\zeta). \tag{17}$$

Again if the boundary is a streamline and satisfies no slip conditions on $|\zeta| = 1$

$$\psi = \frac{\partial \psi}{\partial \bar{\zeta}} = 0, \quad \text{on } |\zeta| = 1, \tag{18}$$

that is

$$-f'(\zeta^{-1}) + z(\zeta)\bar{g}'(\zeta^{-1}) - \bar{z}'(\zeta^{-1})g(\zeta) = 0. \tag{19}$$

If $f(\zeta)$, $g(\zeta)$ are analytic in the fluid region, then

$$\bar{f}'(\bar{\zeta}) = \bar{z}'(\bar{\zeta})g(\bar{\zeta}^{-1}) - z(\bar{\zeta}^{-1})\bar{g}'(\bar{\zeta}), \tag{20}$$

so that

$$2i\psi_{\bar{\zeta}} = \bar{z}'(\bar{\zeta})\{g(\bar{\zeta}^{-1}) - g(\zeta)\} + \{z(\zeta) - z(\bar{\zeta}^{-1})\}\bar{g}'(\bar{\zeta}), \tag{21}$$

or equivalently in terms of the complex fluid velocity

$$u + iv = 2i\psi_{\bar{z}} = g(\bar{\zeta}^{-1}) - g(\zeta) + \frac{\{z(\zeta) - z(\bar{\zeta}^{-1})\}}{\bar{z}'(\bar{\zeta})}\,\bar{g}'(\bar{\zeta}). \tag{22}$$

The stream function ψ can be found by direct integration of (21) as follows:

$$\begin{aligned} 2i\psi = {} & z(\zeta)\bar{g}(\bar{\zeta}) - \bar{z}(\bar{\zeta})g(\zeta) + \int \bar{z}'(\bar{\zeta})g(\bar{\zeta}^{-1})d\bar{\zeta} \\ & - \int z'(\zeta)g(\zeta^{-1})d\zeta + \int \bar{z}(\zeta^{-1})\bar{g}'(\zeta)d\zeta - \int z(\bar{\zeta}^{-1})\bar{g}'(\bar{\zeta})d\bar{\zeta} \end{aligned} \tag{23}$$

and ψ is real with $\psi = 0$, on $|\zeta| = 1$.

It is noted the vorticity is given by

$$wi = \frac{4i\psi_{\zeta\bar{\zeta}}}{z'(\zeta)\bar{z}'(\bar{\zeta})} = \frac{2\{z'(\zeta)\bar{g}'(\bar{\zeta}) - \bar{z}'(\bar{\zeta})g'(\zeta)\}}{z'(\zeta)\bar{z}'(\bar{\zeta})} = 4\,\mathrm{Im}\left\{\frac{z'(\zeta)\bar{g}'(\bar{\zeta})}{z'(\zeta)\bar{z}'(\bar{\zeta})}\right\}. \tag{24}$$

In the present case $z(\zeta) = \zeta - \lambda^2/\zeta$ and a suitable form for $g(\zeta)$ is

$$g(\zeta) = \log(\zeta - d) - \log\left(\zeta - \frac{1}{d}\right) - \frac{A}{d^2}\,\frac{1}{(\zeta - 1/d)}, \tag{25}$$

where the constant

$$A = \frac{(1 - d^2)(1 + \lambda^2)}{d(1 + \lambda^2/d^2)}.$$

The complex fluid velocity is then given by

$$\begin{aligned} u + iv &= 2\psi_{\bar{z}} i \\ &= \log\left(\frac{1 - d\bar{\zeta}}{1 - \bar{\zeta}/d}\right) - \frac{A}{d^2}\,\frac{\zeta}{(1 - \bar{\zeta}/d)} - \log\left(\frac{\zeta - d}{\zeta - 1/d}\right) + \frac{A}{d^2}\,\frac{1}{(\zeta - 1/d)} \\ &\quad + \frac{(\zeta - \lambda^2/\zeta - 1/\bar{\zeta} + \lambda^2\bar{\zeta})}{(1 + \lambda^2/\bar{\zeta}^2)}\left\{\frac{1}{\bar{\zeta} - d} - \frac{1}{\bar{\zeta} - 1/d} + \frac{A}{d^2}\,\frac{1}{(\bar{\zeta} - 1/d)^2}\right\}. \end{aligned} \tag{26}$$

To examine the flow near the boundary it is sufficient to investigate the variation of the vorticity on the boundary. To this end it will suffice to consider

$$\begin{aligned} \operatorname{Im}\left\{z'(\zeta)\,g'\left(\frac{1}{\zeta}\right)\right\} &= \gamma \\ &= \frac{(1 - \lambda^2)(1 - d^2)\sin\phi}{(d^2 + 1 - 2d\cos\phi)} + \frac{A\{\sin 2\phi(d^2 - \lambda^2) - 2d(1 - \lambda^2)\sin\phi\}}{(d^2 + 1 - 2d\cos\phi)^2}. \end{aligned} \tag{27}$$

The vorticity changes sign at $\phi = \alpha$, where

$$\begin{aligned} &(1 - \lambda^2)(1 - d^2)(d^2 + 1 - 2d\cos\alpha) \\ &\quad + 2A[\cos\alpha(d^2 - \lambda^2) - 2d(1 - \lambda^2)] = 0. \end{aligned} \tag{28}$$

To show that a real value of α $(0 < \alpha < \pi)$ exists consider the local sign of γ as $\phi \to 0$, and $\phi \to \pi$. First as $\phi \to 0$

$$\gamma = \phi\left\{\frac{(1 - \lambda^2)(1 - d^2)}{(d - 1)^2} + \frac{2A(\lambda^2 - d^2 - (1 - \lambda^2)d)}{(d - 1)^4}\right\}, \tag{29}$$

which is negative.

Now as ϕ approaches π

$$\gamma = \frac{(\pi - \phi)(d - 1)}{(d + 1)^2(d^2 + \lambda^2)}\left\{2d(d - \lambda^2)(1 + \lambda^2) - (1 - \lambda^2)(1 + d)(d^2 + \lambda^2)\right\}, \tag{30}$$

and choosing a representative value $\lambda^2 = \frac{1}{2}$,

$$\gamma = \frac{(\pi - \phi)(d - 1)}{(d + 1)^2(d^2 + \frac{1}{2})}\left\{3d(d - \tfrac{1}{2}) - \tfrac{1}{2}(1 + d)(d^2 + \tfrac{1}{2})\right\}. \tag{31}$$

The sign of γ thus depends on the sign of

$$3d(2d - 1) - (1 + d)(d^2 + \tfrac{1}{2}). \tag{32}$$

At $d = 1$, this is zero and for values of d slightly greater than unity (32) is positive. For some $\beta > 1$, the quartic vanishes at $d = \beta$, then becomes negative for all larger values of d. Thus as d increases from unity γ is initially negative but becomes positive. It is noted that when $\lambda = 1$ which is the case of a circular arc,

$$\gamma = -(\pi - \phi)2d \frac{(1 + \lambda^2)(1 - d^2)^2}{(1 + 1/d^2)(d + 1)^4} < 0, \tag{33}$$

indicating that a vortex forms on the concave side of the arc. It is now appropriate to discuss the variation of the body profile with d. The body first develops concavity at $\phi = \pi$ and the condition for $\lambda^2 = \frac{1}{2}$ is readily shown to be

$$\frac{9 - \sqrt{77}}{2} < 1 < d < \frac{9 + \sqrt{77}}{2} \tag{34}$$

and for $d > \frac{1}{2}(9 + \sqrt{77})$ the body is everywhere convex to the fluid. The conclusion in this case is that a vortex will be attached to the boundary about $\phi = \pi$ when the body is concave. The fact that (32) is initially negative for values d slightly greater than unity indicates that concavity in itself is not necessarily a condition for a vortex to form except in the case $\lambda = 1$ and the profile is a circular arc. There is no separation when $\lambda = 0$ as this is the case of the profile being a circle, a situation already discussed.

The effect of such a ring vortex is a novel feature for Stokes flow in that attached ring vortex formation is usually associated with convective or Reynolds effect rather than diffusion of vorticity which is the mechanism present here. Since Stokes flow corresponds to a minimum dissipation of energy the vortex should be observable because the motion is stable.

4. Flow past an elliptic limaçon of revolution

In this section an axially symmetric flow past a body with a concave region facing the fluid will be considered briefly. The limaçon $r = 1 + \varepsilon \cos\theta$ contains an indentation about $\theta = \pi$, for $\frac{1}{2} < \varepsilon < 1$, so that this region is concave towards the fluid. Consider a perturbation analysis of the flow, because the exact solution is excessively complicated. The flow may be expanded in the form

$$\psi = \sum_{n=0}^{\infty} \varepsilon^n \psi_n(r, \theta), \tag{35}$$

where the leading term $\psi_0(r, \theta)$ is the Stokes flow past the sphere $r = 1$, given by

$$\psi_0 = \left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \frac{1}{4r}\right)\sin^2\theta. \tag{36}$$

The two term Stokes expansion is readily found by standard methods and is given by

$$\begin{aligned}\psi &= \psi_0 + \varepsilon\psi_1 + O(\varepsilon^2)\\ &= \left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \frac{1}{4r}\right)\sin^2\theta + \tfrac{3}{4}\varepsilon\left(\frac{1}{r^2} - 1\right)\cos\theta\sin^2\theta + O(\varepsilon^2).\end{aligned} \tag{37}$$

The ring vorticity on the boundary is found to be

$$l = \tfrac{3}{2} + 3\varepsilon\cos\theta, \tag{38}$$

so that l first vanishes when $\varepsilon = \frac{1}{2}$, which is the value of ε at which the concave region first develops. The three term Stokes expansion is also found by standard methods and is expressed by

$$\begin{aligned}\psi &= \left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \frac{1}{4r}\right)\sin^2\theta + \tfrac{3}{4}\varepsilon\left(\frac{1}{r^2} - 1\right)\cos\theta\sin^2\theta\\ &\quad + \varepsilon^2(5\cos^2\theta - 1)\left(\frac{3}{10r^3} - \frac{3}{20r}\right)\sin^2\theta + \varepsilon^2\left(\frac{9}{20r} - \frac{3}{10}r\right)\sin^2\theta.\end{aligned} \tag{39}$$

The ring vorticity l in this case is

$$l = \tfrac{3}{2} + 3\varepsilon\cos\theta + \varepsilon^2[-\tfrac{15}{2}\cos^2\theta + \tfrac{3}{2}(5\cos^2\theta - 1) + \tfrac{3}{5}]. \tag{40}$$

At $\theta = \pi$, the ring vorticity vanishes when

$$\tfrac{3}{10}\varepsilon^2 + \varepsilon - \tfrac{1}{2} = 0 \tag{41}$$

giving $\varepsilon \simeq .44$. Since the error is of order ε^3, the indications are that a vortex forms when the limaçon is at first convex everywhere and also when the body is concave, i.e. $\varepsilon > \frac{1}{2}$. This result is subject however to higher order approximations. In any case it seems fairly certain that separation will occur for the limaçon when $\varepsilon > \frac{1}{2}$ and the closed stream surface will leave the surface of the body at a tangent plane. It may be anticipated that other axially symmetric bodies will exhibit a similar phenomenon, but as pointed out in the introduction analytical verification is difficult because conformal mapping techniques do not apply to three dimensional flow with the same simplicity.

REFERENCES

[1] M. Dorrepaal, M. E. O'Neill and K. B. Ranger, Stokes flow past a body with a re-entry region (to be published).

[2] J. H. Michell, On the inversion of plane stress, *Proc. London Math. Soc.*, Vol. 34, 1901–02, p. 142.